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INVARIANT IMBEDDING AND GENERALIZED TRANSPORT THEORY--
A BASIC STOCHASTIC FUNCTIONAL EQUATION

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INVARIANT IMBEDDING AND GENERALIZED TRANSPORT THEORY-- A BASIC STOCHASTIC FUNCTIONAL EQUATION

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1. Introduction. In a series of recent papers, the principle of invariant imbedding has been utilized in the study of a variety of physical processes: radiative transfer,[1,2] (see Preisendorfer[3,4,5]), neutron transport,[6-9] random walk and scattering,[10] and wave propagation.[11]

The aim of the present paper is to extend previous results and techniques so as to include an extensive category of transport processes involving both deterministic and stochastic interaction, general geometries, and the determination of characteristic functions and probabilities as well as fluxes, which is to say, expected values.

Utilizing the principle of invariant imbedding,[1] a basic stochastic functional equation will be derived. From this equation appropriately specialized, can be obtained all the relations pertaining to fluxes contained in the foregoing papers, and in addition corresponding relations for characteristic functions. From these equations for higher moments can be obtained. As far as we know, this use of stochastic functional equations is new.

For intuitive purposes, the reader may use as a model neutron transport theory. In order to emphasize the common features of a number of particle processes, we have encased the problem in a more abstract setting. In this way we obtain an extensive generalization of the fundamental invariance principles of Ambarzumian[12] and Chandrasekhar.[13]

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2. Description of a Generalized Transfer Process. By a 'particle' we shall mean a state vector p specified by a position coordinate x. The state vector contains information regarding energy, direction of motion, and other information required to specify the type of particle. As the particle proceeds through a medium, it engages in interaction of deterministic or stochastic nature with the medium. This interaction leaves the medium unaffected, but is equivalent to a sequence of transformations of the state vector, and in some cases of the number of particles. We shall assume that there are no particle-particle interactions. Although we have begun in another publication[9] the study of collision processes using invariance principles, a general formulation of these more complicated processes is left for another time.

In order to use functional equation techniques, we utilize the concept of a stratified medium. In a finite-dimensional space, conceive of a family of surfaces, of one less dimension, each of which is specified by the parameter x. This family is assumed to have the property that its members can be used to partition the whole space into a denumerable set of strata with the following continuity property. Each stratum possesses the property that the probability of an interaction between a particle and the medium within this stratum can be made arbitrarily small by choosing the stratum to be bounded by arbitrarily closely neighboring members of the family of surfaces. Schematically,

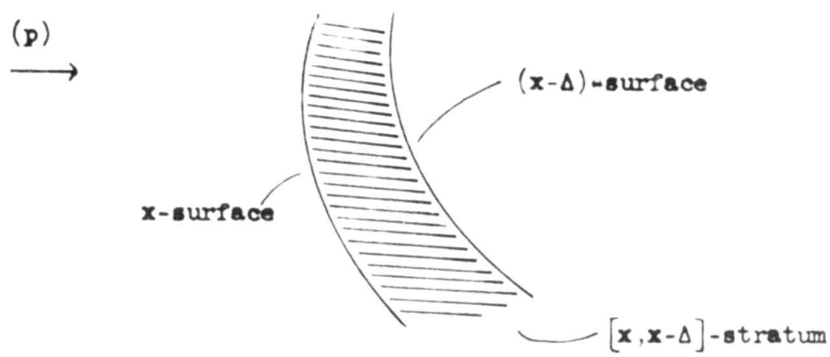

Fig. 1

We shall call the region to the right of an x-surface, an x-region.

The most common stratifications are those accomplished by planes, spheres and cylinders, but more general families of surfaces can equally well be used.

Given a region of space bounded by an x-surface and an 0-surface, we wish to determine the number and nature of the particles existing to the left of the x-surface, 'reflected' particles, and those existing to the right of the 0-surface, 'transmitted' particles, as a result of a particle specified by p incident on the left as in Fig. 1.

Let us now introduce some random variables which we shall use to describe this transport process analytically.

$z(p,q;x)$ = the random number of particles in state q reflected from the x-region over all time, due to an initial particle in state p impinging upon the x-surface from the left at time zero. (2.1)

The direction, velocity, and position on the x-surface are all contained within the state vector.

$r(p;x)$ = 1, if the particle in state p is involved in an interaction in the stratum $[x, x-\Delta]$, or in the stratum $[x-\Delta, x]$. (2.2)
= 0, otherwise.

$s(p;q_1,q_2,\ldots,q_k;x)$ = 1, if the result of an interaction is to produce k particles in states $q_1,q_2,\ldots,q_k$, $k = 1,2,\ldots,$
= 0, otherwise.

By the stochastic variables $z^{(1)}(p,q;x)$, $r^{(1)}(p,x)$, $s^{(1)}(p;q_1,q_2,\ldots,q_k;x)$ we shall mean respectively any of a denumerable set of variables with the properties described above.

Let $T(p,x)$ denote the deterministic change in state caused by passage through the $[x, x-\Delta]$ stratum.

Finally, we make the simplifying assumption that there are only a finite set of possible states. This permits us to deal with characteristic functions rather than characteristic functionals,

and, in any case, is the type of assumption required to carry through any computations.

3. Verbal Description of Process. In order to obtain the functional equation of the following section we view the process in the following fashion. A particle incident upon the x-surface undergoes a deterministic transformation and a stochastic transformation in the $[x, x-\Delta]$ stratum. The result of the deterministic transformation is to convert it from a state p to a state $T(p, x)$. The result of the stochastic transformation is to produce a random number of particles in a random set of states. Each of these particles acts independently of the others as a particle incident upon the $(x-\Delta)$ surface. As a result of a particle incident upon the $(x-\Delta)$ surface, there is produced a random number of emergent particles in a random set of states. Each of these undergoes a deterministic and stochastic transformation as it travels through the $[x-\Delta, x]$ stratum, and so on.

In deriving the basic equations, however, we need not pay attention to particles which are the result of more than one deterministic or stochastic interaction, since in either case the end effects are of order Δ^2. It is this property which enables us to derive relatively simple equations describing quite complicated processes, and it is for this reason that we employ the stratification described above.

4. The Basic Stochastic Functional Equation. Using the invariant imbedding technique and enumerating events we are led to the following stochastic functional relation:

$$
\begin{aligned}
s(p,q;x) = {} & r(p;x) \sum_{\{q_i\}} s(p;q_1,q_2,\dots,q_k;x) \left[\sum_{i=1}^{k} z^{(i)}(q_i, q; x-\Delta) \right] \\
& + (1-r(p;x)) \sum_{i=1}^{i=n} (1-r^{(i)}(q,x)) \quad (n = z(T(p,x)q, x-\Delta)) \\
& + (1-r(p;x)) \sum_{\{q_m\}} \left[\sum_{i=1}^{i=n'} \left[r^{(i)}(q_m;x) \left\{ \sum_{\{v_i\}} s^{(j)}(q_m; v_1, v_2, \dots, v_k; x) \right.\right.\right. \\
& \qquad \left.\left.\left. \cdot \sum_{\ell=1}^{k} z^{(\ell)}(v_\ell, T^{-1}(q), x-\Delta) \right\} \right] \right] \qquad (4.1) \\
& \qquad\qquad (n' = z(T(p,x)q_m, x)) \\
& + y(p,q;x),
\end{aligned}
$$

where $y(p,q;x)$ is a contribution from events that have probability $O(\Delta^2)$ as $\Delta \to 0$. A similar equation can be derived for the neutrons transmitted through the stratum $[x,o]$.

5. Discussion. Taking expected values, we derive the flux equations of the type appearing in the papers cited above. Taking the expected value of $e^{isz(p,q,x)}$, we obtain after a certain amount of analytic manipulation, corresponding functional equations for the characteristic functions. The full results will be presented subsequently.

REFERENCES

1. R. Bellman and R. Kalaba, 'On the Principle of Invariant Imbedding and Propagation through Inhomogeneous Media,' Proc. Nat. Acad Sci. USA, V. 42 (1956), pp. 629-632.

2. R. Bellman and R. Kalaba, 'On the Principle of Invariant Imbedding and Diffuse Reflection from Cylindrical Regions,' Proc. Nat. Acad. Sci. USA, V. 43 (1957), pp. 514-517.

3. R. Preisendorfer, 'Invariant Imbedding for the Principles of Invariance,' Proc. Nat. Acad. Sci. USA, V. 44 (1958), pp. 320-323

4. R. Preisendorfer, 'Functional Relations for the R and T Operators on Plane-Parallel Media,' Proc. Nat. Acad. Sci. USA, V. 44 (1958), pp. 323-328.

5. R. Preisendorfer, 'Time-Dependent Principles of Invariance,' Proc. Nat. Acad. Sci. USA, V. 44 (1958) pp. 328-332.

6. R. Bellman, R. Kalaba, and G. M. Wing, 'On the Principle of Invariant Imbedding and One-Dimensional Neutron Multiplication,' Proc. Nat. Acad. Sci. USA, V. 43 (1957), pp. 517-520.

7. R. Bellman, R. Kalaba and G. M. Wing, 'On the Principle of Invariant Imbedding and Neutron Transport Theory -- I. One-dimensional Case,' Jour. of Math. and Mech., V. 7 (1958), pp. 149-162.

8. R. Bellman, R. Kalaba and G. M. Wing 'Invariant Imbedding and Neutron Transport Theory -- II. Functional Equations,' Jour of Math. and Mech., to appear.

9. R. Bellman, R. Kalaba and G. M. Wing, 'Invariant Imbedding and Neutron Transport Theory -- III. Collision Processes,' Jour of Math. and Mech., to appear.

10. R. Bellman and R. Kalaba, 'Random Walk, Scattering and Invariant Imbedding -- I: One-dimensional Case,' Proc. Nat. Acad Sci USA, V. 43 (1957), pp. 930-933.

11. R. Bellman and R. Kalaba, 'Invariant Imbedding, Wave Propagation and the WKB Approximation,' Proc. Nat. Acad. Sci USA, V. 44 (1958), pp. 317-319.

12. V. Ambarzumian, 'Diffuse Reflection of Light by a Foggy Medium,' Comptes Rendus (Doklady) de l' Academie des Sciences de l' URSS, V. 38 (1943), pp. 229-232.

13. S. Chandrasekhar Radiative Transfer, Oxford, 1950.